"Vector bundles" over quantum Heisenberg manifolds.

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Abstract: We construct, out of Rieffel projections, projections in certain algebras which are strong-Morita equivalent to the quantum Heisenberg manifolds $D^c_{\mu\nu}$. Then, by means of techniques from the Morita equivalence theory, we get finitely generated and projective modules over the algebras $D^c_{\mu\nu}$. This enables us to show that the group $Z + 2\mu Z + 2\nu Z$ is contained in the range of the trace on $K_0(D^c_{\mu\nu})$.

Preliminaries. Let G be the Heisenberg group,

$$G = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in R \right\},$$

and, for a positive integer c, let H_c be the subgroup of G obtained when x, y, and cz are integers. The Heisenberg manifold M_c is the quotient G/H_c.

Non-zero Poisson brackets on M_c that are invariant under the action G on M_c by left translation can be parametrized by two real numbers μ and ν, with $\mu^2 + \nu^2 \neq 0$ ([Rf3]).

For each positive integer c and real numbers μ and ν as above, Rieffel constructed ([Rf3]) a deformation quantization $\{D^{c,\hbar}_{\mu\nu}\}_{\hbar \in R}$ of M_c in the direction of the Poisson bracket $\Lambda_{\mu\nu}$.

Since $D^{c,\hbar}_{\mu\nu}$ is isomorphic to $D^{c,1}_{\hbar\mu,\hbar\nu}$, and we will not need to keep track of the Planck constant $\hbar$, we absorb it from now on into the parameters μ and ν. Thus we will use $D^c_{\mu\nu}$ to denote $D^{c,1}_{\mu\nu}$.

As shown in [Rf3], the algebra $D^c_{\mu\nu}$ can be described as the generalized fixed-point algebra of the crossed-product $C_0(R \times T) \times_\lambda Z$, where $\lambda_p(x, y) = (x + 2p\mu, y + 2p\nu)$, for all $p \in Z$, under the action ρ of Z defined by

$$(\rho_k \Phi)(x, y, p) = e(ckp(y - p\nu))\Phi(x + k, y, p),$$

where k, $p \in Z$, $\Phi \in C_c(R \times T \times Z)$, and, for any real number x, $e(x) = exp(2\pi i x)$.

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The action ρ defined above corresponds to the action ρ defined in [Rf3, p.539], after taking Fourier transform in the third variable to get the algebra denoted in that paper by $A_\hbar$, and viewing $A_\hbar$ as a dense *-subalgebra of $C_0(R \times T) \times_\lambda Z$ via the embedding J defined in [Rf3, p.547]. Equivalently, $D^c_{\mu\nu}$ is the closure in the multiplier algebra of $C_0(R \times T) \times_\lambda Z$ of the *-subalgebra D_0 consisting of functions $\Phi \in C(R \times T \times Z)$ which have compact support on Z and satisfy

$$\Phi(x + k, y, p) = e(-ckp(y - p\nu))\Phi(x, y, p),$$

for all $k, p \in Z$, and $(x, y) \in R \times T$ (D_0 is the image under the embedding J mentioned above of the subalgebra denoted by C^ρ in the proof of [Rf3, Thm.5.4]).

There is a faithful trace ([Rf3]) τ_D on $D^c_{\mu\nu}$ defined for $\Phi \in D_0$, by

$$\tau_D(\Phi) = \int_{T^2} \Phi(x, y, 0)dxdy.$$

It can be shown ([Ab2]) that the algebra $D^c_{\mu\nu}$ is strong-Morita equivalent to the generalized fixed-point algebra $E^c_{\mu\nu}$ of the crossed product $C_0(R\times T)\times_\sigma Z$ under the action γ of Z, where $\sigma_k(x, y) = (x - k, y)$ and

$$(\gamma_p\Phi)(x, y, k) = e(-ckp(y - p\nu))\Phi(x - 2p\mu, y - 2p\nu, k),$$

for k, $p \in Z$ and $\Phi \in C_c(R \times T \times Z)$.

As for the quantum Heisenberg manifolds case, $E^c_{\mu\nu}$ can also be described (see [Ab2]) as the closure in the multiplier algebra of $C_0(R \times T) \times_\sigma Z$ of the *-algebra E_0 consisting of functions $\Phi \in C(R \times T \times Z)$, with compact support on Z and such that

$$\Phi(x - 2p\mu, y - 2p\nu, k) = e(ckp(y - p\nu))\Phi(x, y, k),$$

for all $k, p \in Z$, $(x, y) \in R \times T$. The equivalence $D^c_{\mu\nu}$-$E^c_{\mu\nu}$ bimodule X constructed in [Ab2] is the completion of $C_c(R \times T)$ with respect to either one of the norms induced by the $D^c_{\mu\nu}$ and $E^c_{\mu\nu}$-valued inner products given by

$$< f, g >_D (x, y, p) = \sum_{k\in Z} e(ckp(y - p\nu))f(x + k, y)\overline{g}(x - 2p\mu + k, y - 2p\nu)$$

and

$$< f, g >_E (x, y, k) = \sum_{p\in Z} e(-ckp(y-p\nu))\overline{f}(x-2p\mu, y-2p\nu)g(x-2p\mu+k, y-2p\nu),$$

respectively, where $f, g \in C_0(R \times T)$, $\Phi \in D_0$, $\Psi \in E_0$, $(x, y) \in R \times T$, and $k, p \in Z$.

In what follows we produce finitely generated and projective modules over the algebras $D^c_{\mu\nu}$. To do this we apply to the Morita equivalence structure described above the methods for constructing projections provided by the Morita equivalence theory. Finally, we get a partial description of the range of the trace at the level of $K_0(D^c_{\mu\nu})$.

Remark 1 *First notice that both D_0 and E_0 have identity elements I_D and I_E, respectively, defined by*

$$I_D(x, y, p) = \delta_0(p) \text{ and } I_E(x, y, k) = \delta_0(k),$$

for $(x, y) \in R \times T$ and $k, p \in Z$.

Therefore, by [Rf2, Prop. 1.2], if P is a projection in E_0, then XP is a projective finitely generated left module over $D^c_{\mu\nu}$, and the corresponding projection in $M_m(D^c_{\mu\nu})$ is given by

$$Q = \begin{pmatrix} < y_1, x_1 >_D & ... & < y_m, x_1 >_D \\ ... & ... & ... \\ < y_1, x_m >_D & ... & < y_m, x_m >_D \end{pmatrix}$$

where, for $i = 1, ..., m$, $x_i, y_i \in XP$ are such that $P = \sum_{i=1}^{i=m} < x_i, y_i >_E$.

On the other hand ([Rf1, Prop. 2.2]), the trace τ_D on $D^c_{\mu\nu}$ induces a trace τ_E on $E^c_{\mu\nu}$ via

$$\tau_E(< f, g >_E) = \tau_D(< g, f >_D).$$

A straightforward computation shows that for $\Psi \in E_0$ we have

$$\tau_E(\Psi) = \int_0^{2\mu} \int_0^1 \Psi(x, y, 0) dx dy.$$

Then, in the notation above we get

$$\tau_D(Q) = \sum_{i=1}^{1=m} \tau_D(< y_i, x_i >_D) = \sum_{i=1}^{i=m} \tau_E(< x_i, y_i >_E) = \tau_E(P).$$

Theorem 1 *The bimodule X is a finitely generated and projective $D^c_{\mu\nu}$-module of trace 2μ. If $\nu \in [0, 1/2]$, and $\mu > 1$, then there is a finitely generated projective $D^c_{\mu\nu}$-submodule of X with trace 2ν.*

Proof:

Let us take $P = I_E$, in the notation of Remark 1. Then $X = XP$ is finitely generated and projective and its trace is $\tau_E(I_E) = 2\mu$.

We now find a projection P in E_0 with $\tau_E(P) = 2\nu$, when $\nu \in [0, 1/2]$ and $\mu > 1$, which ends the proof, in view of Remark 1.
So let us consider self-adjoint elements P of the form:

$$P(x, y, p) = f(x, y)\delta_1(p) + h(x, y)\delta_0(p) + \overline{f}(x - 1, y)\delta_{-1}(p),$$

where h and f are bounded functions on $R \times T$ and h is real-valued. Our next step is to get functions f and h such that P is a projection in $E^c_{\mu\nu}$.

Now,

$$(P * P)(x, y, p) = \sum_{q \in Z} P(x, y, q)P(x + q, y, p - q),$$

and it follows that $P * P = P$ if and only if, for all $(x, y) \in R \times T$:

1) $f(x, y)f(x + 1, y) = 0$

2) $f(x, y)[h(x + 1, y) + h(x, y) - 1] = 0$

3) $|f(x, y)|^2 + |f(x - 1, y)|^2 = h(x, y)(1 - h(x, y))$.

We also want P to be in E_0, so we require

$$P(x, y, p) = e(cp(y + \nu))P(x + 2\mu, y + 2\nu, p), \text{ that is}$$

4) $f(x, y) = e(c(y + \nu))f(x + 2\mu, y + 2\nu)$

and

5) $h(x, y) = h(x + 2\mu, y + 2\nu)$.

It was shown on [Rf1, 1.1] that for any $\zeta \in [0, 1/2]$ there are maps $F, H \in C(T)$ such that:

1)' $F(t)F(t - \zeta) = 0$

2)' $F(t)[1 - H(t) - H(t - \zeta)] = 0$

3)' $H(t)[1 - H(t)] = |F(t)|^2 + |F(t + \zeta)|^2$

4)' $\int_T H = \zeta$

5)’ $0 \leq H(t) \leq 1$ for any $t \in T$ and F vanishes on $[1/2, 1]$.

Let us assume that $\nu \in [0, 1/2]$, $\mu > 1$ and let F and H be functions satisfying 1)’-5)’ for $\zeta = \nu/\mu$.

Translation of t by ζ in equations 1)’-5)’ plays the same role as translation of x by 1 in equations 1)-5), which suggests taking ζx as the variable t. However, the variable y will play an important role in getting f and h to satisfy 4) and 5), for which we need to take $t = 1/2 + y - \zeta x$.
So let

$$h(x, y) = H(1/2 + y - \zeta x),$$

so h is in $C(R \times T)$, and it is real-valued and bounded.
Also,

$$h(x + 2\mu, y + 2\nu) = H(1/2 + y + 2\nu - \zeta x - 2\nu) = H(1/2 + y - \zeta x) = h(x, y),$$

so h satisfies 5).
Now, for $(x, y) \in [0, 2\mu] \times [0, 1]$, set

$$f(x, y) = \begin{cases} F(1/2 + y - \zeta x) & \text{if } y \leq x/(2\mu) \\ e(c(y + \nu))F(1/2 + y - \zeta x) & \text{if } y \geq x/(2\mu) \end{cases}$$

To show f is continuous it suffices to prove that $F(1/2 + y - \zeta x) = 0$ when $y = x/(2\mu)$, and that follows from the fact that F vanishes on $[1/2, 1]$, and from the conditions on μ and ν.
Now, since $f(x, 1) = f(x, 0)$, f is continuous on $[0, 2\mu] \times T$. We want to extend f to $R \times T$ by letting

$$f(x + 2\mu, y) = e(-c(y - \nu))f(x, y - 2\nu),$$

so as to have f satisfy 4). We only need to show that

$$f(2\mu, y) = e(-c(y - \nu))f(0, y - 2\nu) \text{ for any } y \in T.$$

For an arbitrary $y \in R$, let $k, l \in Z$ be such that $y + k, y - 2\nu + l \in [0, 1]$. Then,

$$f(2\mu, y) = F(1/2 + y + k - 2\nu) = F(1/2 + y - 2\nu), \text{ and}$$

$$f(0, y - 2\nu) = f(0, y - 2\nu + l) = e(c(y - \nu + l))F(1/2 + y - 2\nu) =$$
$$= e(c(y - \nu))f(2\mu, y),$$

as wanted, and f, extended to $R \times T$ as above, satisfies 4). It remains to show that f and g satisfy 1), 2) and 3):

1) $|f(x,y)f(x+1,y)| = |F(1/2+y-\zeta x)F(1/2+y-\zeta x-\zeta| = 0$, by 1)'.

2) $|f(x,y)[h(x+1,y)+h(x,y)-1]| = |F(1/2+y-\zeta x)[H(1/2+y-\zeta x-\zeta)+H(1/2+y-\zeta x)-1]| = 0$, by 2)'.

3) $|f(x,y)|^2+|f(x-1,y)|^2 = |F(1/2+y-\zeta x)|^2+|F(1/2+y-\zeta x+\zeta)|^2 = = H(1/2+y-\zeta x)[1-H(1/2+y-\zeta x)] = h(x,y)(1-h(x,y)),$ by 3)'.

Therefore P is a projection on E_0. Besides,

$$\tau_E(P) = \int_0^{2\mu}\int_T h(x,y)dydx =$$

$$\int_0^{2\mu}(\int_T H(1/2+y-\zeta x)dy)dx = \int_0^{2\mu}\zeta = 2\mu\zeta = 2\nu, \text{ by 5)'}.$$

Q.E.D.

The following propositions enable us to extend the previous results by dropping the restrictions on μ and ν.

Notation: In Propositions 1 and 2 Π denotes the faithful representation of $D^c_{\mu\nu}$ on $L^2(R \times T \times Z)$ obtained by restriction of the left regular representation of the multiplier algebra of $C_0(R \times T) \times_\lambda Z$ on $L^2(R \times T \times Z)$, i.e.

$$(\Pi_\Phi\xi)(x,y,p) = \sum_{q\in Z}\Phi(x+2p\mu, y+2p\nu, q)\xi(x,y,p-q),$$

for $\Phi \in D_0$, $\xi \in L^2(R \times T \times Z)$, and $(x,y,p) \in R \times T \times Z$.

Notice that Π is faithful because Z is amenable ([Pd, 7.7.5 and 7.7.7.]).

Proposition 1 *There is a trace-preserving isomorphism between $D^c_{\mu\nu}$ and $D^c_{\mu+k,\nu+l}$, for all $k,l \in Z$.*

<u>*Proof*</u>:

It is clear that $\Phi \mapsto \Phi$ is an isomorphism between $D^c_{\mu\nu}$ and $D^c_{\mu,\nu+l}$, so let us assume $l=0, k=1$.

Let $J : D^c_{\mu+1,\nu} \longrightarrow D^c_{\mu\nu}$ be defined at the level of functions in D_0 by:

$$(J\Phi)(x,y,p) = e(c(4p^3\nu/3 - p^2y))\Phi(x,y,p).$$

It is easily checked that $J\Phi \in D^c_{\mu\nu}$ for all $\Phi \in D^c_{\mu+1,\nu}$. Besides, the unitary operator $U : L^2(R \times T \times Z) \longrightarrow L^2(R \times T \times Z)$ given by

$$U\xi(x,y,p) = e(c(-4p^3\nu/3 - p^2 y))\xi(x,y,p)$$

intertwines $\Pi_{J\Phi}$ and Π_Φ:

$$(\Pi_\Phi U\xi)(x,y,p) = \sum_{q\in Z} \Phi(x+2p(\mu+1), y+2p\nu, q)U\xi(x,y,p-q) =$$

$$= \sum_{q\in Z} e(-2pcq(y+(2p-q)\nu)e(c((-4\nu/3)(p-q)^3 - (p-q)^2 y))\cdot$$

$$\cdot\Phi(x+2p\mu, y+2p\nu, q)\xi(x,y,p-q) =$$

$$= e(c(-4\nu p^3/3 - p^2 y)) \sum_{q\in Z} e(c(4q^3\nu/3 - q^2(y+2p\nu))\Phi(x+2p\mu, y+2p\nu, q)\xi(x,y,p-q) =$$

$$= (U\Pi_{J\Phi}\xi)(x,y,p).$$

Also,

$$\tau(J\Phi) = \int_0^1 \int_T J\Phi(x,y,0) = \int_0^1 \Phi(x,y,0) = \tau(\Phi).$$

Q.E.D.

Proposition 2 *There is a trace-preserving isomorphism between* $D^c_{\mu\nu}$ *and* $D^c_{-\mu,-\nu}$.

Proof:

Let $J : D^c_{\mu\nu} \longrightarrow D^c_{-\mu,-\nu}$ be defined, at the level of functions, by:

$$(J\Phi)(x,y,p) = \Phi(-x,-y,p).$$

It is easily checked that $J\Phi \in D_{-\mu,-\nu}$. Besides, the unitary operator $U : L^2(R \times T \times Z) \longrightarrow L^2(R \times T \times Z)$ defined by

$$(U\xi)(x,y,p) = \xi(-x,-y,p)$$

intertwines Π_Φ and $\Pi_{J\Phi}$:

$$[\Pi_{J\Phi}(U\xi)](x,y,p) = \sum_{q\in Z} (J\Phi)(x-2p\mu, y-2p\nu, q)\xi(-x,-y,p-q) =$$

$$= \sum_{q \in Z} \Phi(-x + 2p\mu, -y + 2p\nu, q)\xi(-x, -y, p - q) =$$

$$= (\Pi_\Phi \xi)(-x, -y, p) = (U\Pi_\Phi \xi)(x, y, p).$$

Finally, J preserves the trace:

$$\tau(J\Phi) = \int_{T^2} \Phi(-x, -y, 0) = \tau(\Phi).$$

Q.E.D.

Theorem 2 *The range of the trace on $K_0(D^c_{\mu\nu})$ contains the set $Z + 2\mu Z + 2\nu Z$.*

Proof:

We obviously have $Z \subseteq \tau(K_0(D^c_{\mu\nu}))$, since $D^c_{\mu\nu}$ has an identity element. Besides, it follows from Theorem 1 that $2\mu Z \subseteq \tau(K_0(D^c_{\mu\nu}))$. So it only remains to show that $2\nu Z \subseteq \tau(K_0(D^c_{\mu\nu}))$.

Let $k \in Z$ be such that $\nu' = \pm\nu + k$ and $\nu' \in [0, 1/2]$. Then one can find $l \in Z$ and $\mu' = \pm\mu + l$ such that $\mu' \geq 1$. Thus , by Propositions 1 and 2 we have that $\tau(K_0(D^c_{\mu'\nu'})) = \tau(K_0(D^c_{\mu\nu}))$.

Now, by Theorem 1 there is a projection in $M_m(D^c_{\mu'\nu'})$ for some positive integer m with trace $2\nu' = \pm 2\nu + 2k$, which ends the proof.

Q.E.D.

Remark. It can be shown ([Ab1]) that the inclusion in the previous theorem is actually an identity.

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